Range ,Rank and Nullity of Linear Transformation

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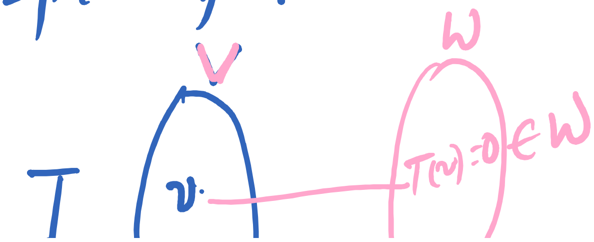
Range :- If $V(F)$ and $W(F)$ are vector spaces and $T: V \to W$ is a linear transformation, Then the image set of V under T is called Range of T. which is denoted as Range T or Image T or $R(T)$ or $T(V)$

i.e $\text{Range } T = \{T(v) : v \in V\}$

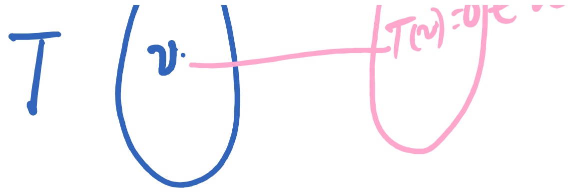
Range T is also called Range space.

Null Space :- (or Kernel) :- If $V(F)$ and $W(F)$ are two vector spaces and $T: V \to W$ is a linear transformation then the set of all those vectors in V whose image under T is zero, is called Kernel or Null space of T which is denoted by $N(T)$ i.e

$$\text{Null Space of } T = N(T) = \{v \in V : T(v) = 0 \in W\}$$



$v \in N(T)$



Rank:- If $V(F)$ and $W(F)$ be vector spaces and $T: V \to W$ be a linear transformation, then the dimension of the range space of T is called the Rank of T and it is denoted as $\rho(T)$

Thus $\rho(T) = \dim(\text{Range } T)$

Nullity:- If $V(F)$ and $W(F)$ be vector space and $T: V \to W$ be a L.T. then the dimension of null space of T is called nullity of T and it is denoted as $\nu(T)$.

i.e $\nu(T) = \dim(\text{Null Space of } T)$

Rank - Nullity Theorem
(Sylvester's Law of nullity)

If $V(F)$ and $W(F)$ are vector spaces and $T: V \to W$ is a linear transformation. suppose V is of is a linear transformation. Suppose V is of dimension n (i.e V is finite dimensional vector space) Then

$$\boxed{\text{Rank } T + \text{Nullity } T = \dim V.}$$

Q:- For a linear transformation $T: V \to W$, find the basis and dimension of its

1. Range space
2. Null space

and Also verify $R(T) + \text{Nullity}(T) = \dim V$.

i.e Rank - Nullity theorem.

(a) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x+y, x-y, y)$

Sol: St. Basis of $\mathbb{R}^2 = \{e_1, e_2\} = \{(1,0), (0,1)\}$

First of all we shall find basis for Range T.

Since $B = \{(1,0), (0,1)\}$ is the basis of $\mathbb{R}^2$

Since $B = \{(1,0), (0,1)\}$
$\qquad\qquad\quad \downarrow \qquad \downarrow$
$\qquad\qquad\quad e_1 \qquad e_2$

$B_1 = \{T(e_1), T(e_2)\}$ generates Range T

$T(e_1) = T(1,0) = (1+0, 1-0, 0) = (1,1,0)$

$T(e_2) = T(0,1) = (0+1, 0-1, 1) = (1,-1,1)$

$[\,T(x,y) = (x+y, x-y, y)$

Now To find basis of Range T,
we want to show $\{T(e_1), T(e_2)\}$ is basis set.

for that Purpose firstly we show $T(e_1)$ and $T(e_2)$ are L.I.

Consider $\alpha\,(T(e_1)) + \beta\, T(e_2) = 0$

$$\alpha\,(1,1,0) + \beta\,(1,-1,1) = (0,0,0)$$

$$(\alpha+\beta,\ \alpha-\beta,\ \beta) = (0,0,0)$$

$$\therefore\ \alpha+\beta = 0$$
$$\alpha-\beta = 0$$
$$\beta = 0$$

$\therefore \boxed{\beta = 0}$ and also $\boxed{\alpha = 0}$

$\therefore$ $(1, 1, 0)$ and $(1, -1, 1)$ are L.I vectors.

$\Rightarrow$ $B_1 = \{(1, 1, 0), (1, -1, 1)\}$ is L.I set.

$\Rightarrow$ Range space of $T = \{(1, 1, 0), (1, -1, 1)\}$

$\therefore$ Rank of T = Number of elements in B_1

$= 2$.

Now we find out basis of Null space of T.

let $v = (x, y) \in N(T)$

$T(v) = T(x, y) = 0$

$\Rightarrow (x+y, x-y, y) = (0, 0, 0)$

$\Rightarrow$ $x+y = 0$

$x - y = 0$

$y = 0$

$\boxed{x = 0}$, $\boxed{y = 0}$

$v = (x, y) = (0, 0)$

$(1, 0)$

$(0, 1)$

$(1, 1)$

So that $v \in N(T) \Rightarrow v = (0, 0) = 0$

$\therefore$ Null space of $T = (0)$

Nullity T = dim (Null Space).
$= 0$

Thus Nullity T + Rank T $= 0+2$
$= 2 = \dim(R^2)$
$= \dim V.$

verified the Rank Nullity Theorem

Q:- verify Rank-Nullity Theorem for the Transformation $T: R^3 \to R^3$ defined by
$T(x,y,z) = (x+2y,\ y-z,\ x+2z)$

or,

for a linear transformation $T: \underset{V}{R^3} \to R^3$ defined by
$T(x,y,z) = (x+2y,\ y-z,\ x+2z)$

find (i) its Range Space
(ii) its Null Space
also verify Rank(T) + Nullity(T) = dim V

Note:: General Basis of $R^2 = \{\underset{e_1}{(1,0)},\ \underset{e_2}{(0,1)}\} = \{e_1, e_2\}$
$R^3 = \{\underset{e_1}{(1,0,0)},\ \underset{e_2}{(0,1,0)},\ \underset{e_3}{(0,0,1)}\}$

Note:: General Basis of $R^2 = \{(1,0), (0,1)\} = \{e_1, e_2\}$

$R^3 = \{(1, \overset{e_1}{0}, 0), (0, \overset{e_2}{1}, 0), (0, 0, \overset{e_3}{1})\} = \{e_1, e_2, e_3\}$

$R^4 = \{(1, 0, \overset{e_1}{0}, 0), (0, 1, \overset{e_2}{0}, 0), (0, \overset{e_3}{0}, 1, 0), (0, 0, \overset{e_4}{0}, 1)\}$
$= \{e_1, e_2, e_3, e_4\}$

Sol:: We know that a basis for R^3 is

$$B = \{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

(i) first of all we shall find Basis for Range T.

$\because$ B is basis of R^3

$\therefore$ $B_1 = \{T(e_1), T(e_2), T(e_3)\}$ generates Range T.

$T(x,y,z) = (x+2y, y-z, x+2z)$

Here $T(e_1) = T(1,0,0) = (1+0, 0-0, 1+0) = (1,0,1)$

$T(e_2) = T(0,1,0) = (0+2, 1-0, 0+0) = (2,1,0)$

$T(e_3) = T(0,0,1) = (0+0, 0-1, 0+2) = (0,-1,2)$

$\therefore$ $B_1 = \{(1,0,1), (2,1,0), (0,-1,2)\}$ generates Range T.

To find basis for range T, we have to find out L.I. vectors from $T(e_1)$, $T(e_2)$, $T(e_3)$

... the matrix and reduce

L.I. vectors from ...

for this consider the matrix and reduce it to echelon form

i.e $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $(1, 0, 1)$ $(0, 1, -2)$ form L.I set of vectors which generates Range T.

$\therefore$ Range space of $T = \{ \underline{(1,0,1)}, \underline{(0,1,-2)} \}$

$\therefore$ Rank $T =$ dim(Range space) $= 2$.

2nd part: To find basis for Null space.

let $v = (x, y, z) \in N(T)$

$T(v) = T(x,y,z) = 0$

$(x+2y,\ y-z,\ x+2z) = (0,0,0)$

$$\Rightarrow \left.\begin{matrix} x+2y = 0 \\ y-z = 0 \\ x+2z = 0 \end{matrix}\right] \text{———} (1)$$

To find the basis of Null Space of T

consider matrix $P = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

$$R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ x + 2y = 0 \Rightarrow x = -2y$

$y - z = 0 \qquad y = z$

solution set is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} y$

$\therefore \{(-2,1,1)\}$ is a basis for

Hence $B_2 = \{(-2, 1, 1)\}$ is a basis for null space of T.

$\dim(N(T)) = 1$

$\Rightarrow$ Nullity of $T = \dim(N(T)) = 1$

$\therefore$ Rank(T) + Nullity of $T = 2+1 = 3 = \dim \mathbb{R}^3$
$= \dim V.$

verified the Rank-Nullity Theorem

N.K. Jha
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